Optimal Evasion with a Path-Angle Constraint and Against Two Pursuers

Joseph Ben-Asher,* Eugene M. Cliff,† and Henry J. Kelley‡
Virginia Polytechnic Institute and State University, Blacksburg, Virginia

We consider the problem of optimal evasion when the pursuer is known to employ fixed-gain proportional navigation. The performance index is a measure of closest approach. The analysis is done for planar motions at constant speed, and the kinematics are first linearized around a nominal collision course. Two cases are studied. The first involves optimal evasion with a terminal path-angle constraint for the evader, and the second, optimal evasion against more than one pursuer. The latter is studied as an optimal evasion against one pursuer with state constraints imposed by the others. The optimal controls are shown to be "bang-bang," with the number of switches depending on the pursuer's navigation gain and the particular constraints of each case.

Introduction

PURSUIT-EVASION problems have been traditionally classified among the classical examples of differential game theory. In the last 15 years, a different approach has been applied to these problems,[1-4] namely, to fix the pursuer's strategy and form a one-sided optimal control problem for the evader. This approach, being conceptually simpler than the former, enables more realistic models to be applied for the dynamics of the opponents. In general, the fixed pursuer's strategy has been taken as constant gain, proportional navigation that, under some formulations, is an optimal strategy for the pursuer.

We shall consider two related problems in which the optimal evasive maneuvers are subjected to some state constraints. The first problem to be considered in this work is the optimal evasion with a terminal path-angle constraint. The motivation for this problem is derived from cases in which the evading vehicle is a missile, guiding toward a fixed target, and the pursuer is an interceptor trying to protect this target. Thus, the evasive maneuver is constrained by a terminal path- (or heading) angle constraint to guarantee capture of the fixed target. The interceptor is successful if the evader is destroyed, or if it is made to deviate significantly from its course. It will be shown that this problem has some interesting features whose importance may exceed the bounds of the pursuit-evasion conflict.

The motivation for the second problem is self-evident. We shall consider the problem of optimal evasion against more than one, starting with two pursuers. This problem will be studied as an optimal evasion against one pursuer subject to a minimum required miss distance of the others. We shall allow the pursuers to select their launch time so as to minimize the closest approach.

The models to be used are all linear, due to the relative complexity of the problems. Consequently, the results will be valid in the vicinity of the nominal collision courses.

Basic Assumptions and Equations

We shall make the following assumptions:[4]

1) The pursuit-evasion conflict is two-dimensional, in the horizontal plane.
2) The speeds of the pursuer P and the evader E are constant.
3) The trajectories of P and E can be linearized around their collision triangle.
4) P applies a fixed-gain proportional navigation.
5) E has complete information on P's system and the collision course.
6) P's acceleration is subject to a first-order lag.
7) E's lateral acceleration is bounded, P's unbounded.

Referring to Fig. 1, by assumptions 1-3 we obtain the following equations:

$$\sin(\gamma_{e_o} + \gamma_e) = \sin(\gamma_{e_o}) + \cos(\gamma_{e_o})\gamma_e \tag{1}$$

$$\dot{R} = -V_r = V_p \cos(\gamma_{p_o}) - V_e \cos(\gamma_{e_o}) = \text{const} \tag{2}$$

and

$$\dot{y} = \dot{y}_e - \dot{y}_p = V_e \cos(\gamma_{e_o})\gamma_e - V_p \cos(\gamma_{p_o})\gamma_p \tag{3}$$

Since the nominal course leads to collision, we have the relation

$$V_e \sin(\gamma_{e_o}) - V_p \sin(\gamma_{p_o}) = 0 \tag{4}$$

P's commanded acceleration is by assumption 4:

$$\ddot{y}_{p_c} = N'V_r\dot{\sigma}, \qquad N' = N(V_p/V_r)\cos\gamma_{p_o} \tag{5}$$

where the line-of-sight (LOS) rate is determined by

$$\dot{\sigma} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{R}\right) = \frac{y}{V_r(t_f - t)^2} + \frac{\dot{y}}{V_r(t_f - t)} \tag{6}$$

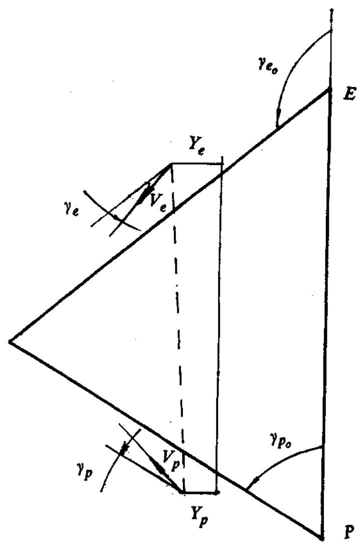

Fig. 1 Problem geometry.

Received April 15, 1987; revision received Oct. 26, 1987.

*Graduate Student, Department of Aerospace and Ocean Engineering. Student Member AIAA.

†Professor, Department of Aerospace and Ocean Engineering. Member AIAA.

‡Christopher Kraft Professor, Department of Aerospace and Ocean Engineering. Fellow AIAA.

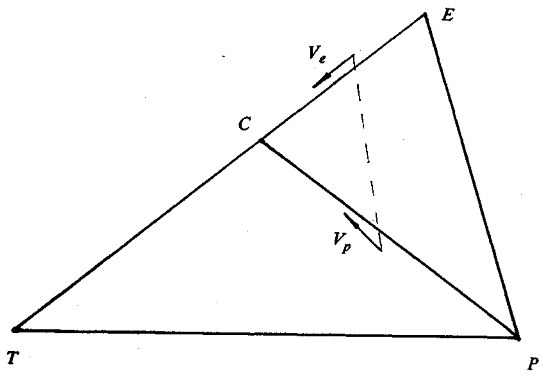

Fig. 2 Constrained problem geometry.

E is free to choose its required acceleration, based on the information available to it by assumption 5.

The dynamics of P are by assumption 6:

$$\frac{\mathrm{d}\ddot{y}_p}{\mathrm{d}t} = \frac{\ddot{y}_{p_c} - \ddot{y}_p}{\tau_p} \tag{7}$$

where $\ddot{y}_{p_c}$ is P's commanded acceleration.

Optimal Evasion with a Path-Angle Constraint

Problem Formulation

Figure 2 presents the relative geometry of the problem. T is a fixed target to which E is homing. P is an interceptor that defends T. Therefore, E's mission, in general, is to first evade from P and then to destroy T. If, however, EC is larger than ET (C is the collision point), then the evasion problem is irrelevant.

In general, the heading error of E's velocity vector from its line of sight to T will determine the miss of the ground target homing mission. Thus, our linear model implies a terminal path- (or heading) angle constraint on the optimal evasion problem.

The state vector for this case is (if we follow Ref. 4)

$$x = \mathrm{col}(\gamma_e, y, \gamma_p, \dot{\gamma}_p) \tag{8}$$

and the control u is

$$u = \frac{\dot{\gamma}_e}{\dot{\gamma}_{e_m}} \tag{9}$$

where

$$\dot{\gamma}_{e_m} = |\dot{\gamma}_e|_{\max} \tag{10}$$

The linear state equations relating x to u are

$$\dot{x}_1 = \dot{\gamma}_{e_m} u \tag{11}$$

$$\dot{x}_2 = V'_e x_1 - V'_p x_3 \tag{12}$$

$$\dot{x}_3 = x_4 \tag{13}$$

$$\dot{x}_4 = \frac{N'V'_e}{\tau_p V'_p(t_f - t)} x_1 + \frac{N'}{\tau_p V'_p(t_f - t)^2} x_2 - \frac{N'}{\tau_p(t_f - t)} x_3 - \frac{1}{\tau_p} x_4 \tag{14}$$

where

$$V'_e = V_e \cos(\gamma_{e_o}) \tag{15}$$

$$V'_p = V_p \cos(\gamma_{p_o}) \tag{16}$$

The optimal problem therefore is to find u that minimizes the payoff $J = -x_2^2(t_f)$ subject to $|u(t)| \le 1$ and $|\gamma_e(t_f)| \le \gamma_f$. The terminal time is fixed and is determined by the nominal collision time of P and E. At this time, the state equations become singular since the time to go is in the denominator of Eq. (14). This expresses the fact that, by proportional navigation, the pursuer-required accelerations go to infinity at the terminal state. This singularity should be taken into account in the computational procedure.

Problem Analysis

We shall define the Hamiltonian

$$H(\lambda, x, u) = \dot{x}_1\lambda_1 + \dot{x}_2\lambda_2 + \dot{x}_3\lambda_3 + \dot{x}_4\lambda_4 \tag{17}$$

The adjoint variables should satisfy the following equations:

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \tag{18}$$

The transversality conditions for this problem are[5]

$$\lambda_i(t_f) = 0 \qquad \text{for } i = 3, 4 \tag{19}$$

$$\lambda_2(t_f) = -2x_2(t_f) \tag{20}$$

$$\lambda_1(t_f) = \begin{cases} v \ge 0 & \text{if} \quad \gamma_e(t_f) = \gamma_f \\ 0 & \text{if} \quad -\gamma_f < \gamma_e(t_f) < \gamma_f \\ v \le 0 & \text{if} \quad \gamma_e(t_f) = -\gamma_f \end{cases} \tag{21}$$

The last condition is a Kuhn–Tucker-like condition for our inequality end-constraint. The variation of the payoff due to the terminal path-angle variation is $-\lambda_1(t_f)\delta x_1(t_f)$. The transversality condition implies that violating the constraint would yield a better payoff.

From the maximum principle, we get

$$u(t) = -\mathrm{sgn}\lambda_1(t) \tag{22}$$

Hence λ_1 is the control switching function. Thus, the solution is a bang-bang type of control for E's actual acceleration. Since the adjoint equations are independent of the state equations, except through the boundary conditions, the switching function and any finite time derivative of it do not contain the control explicitly. Therefore, no finite-order singular subarcs are possible.

The problem without the path-angle constraint was solved analytically in Ref. 4 by Laplace transforms. For this problem, $\lambda_1(t_f) = 0$. Let $\theta = (t_f - t)/\tau_p$ (normalized time to go). The adjoint equations are

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}\theta} = (\tau_p V'_e)\lambda_2 + \left(\frac{N'V'_e}{\tau_p V'_p \theta}\right)\lambda_4 \tag{23}$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}\theta} = \left(\frac{N'}{\tau_p^2 V'_p \theta^2}\right)\lambda_4 \tag{24}$$

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}\theta} = -(\tau_p V'_p)\lambda_2 - \left(\frac{N'}{\tau_p \theta}\right)\lambda_4 \tag{25}$$

$$\frac{\mathrm{d}\lambda_4}{\mathrm{d}\theta} = \tau_p\lambda_3 - \lambda_4 \tag{26}$$

The solution for $\lambda_1(\theta)$ has the form

$$\hat{\lambda}_1(\theta) = p(\theta)\exp(-\theta) \tag{27}$$

where p is a polynomial of order $N' - 1$ (for an integer effective gain N') with one root at the origin. Consequently, there are $N' - 2$ switching points. (We do not consider the final time as a switching point.) The more general result (for any N', integer or not) is the inverse Laplace transform of

$$\hat{\Lambda}_1(s) = c\frac{s^{N'-2}}{(s+1)^{N'}} \tag{28}$$

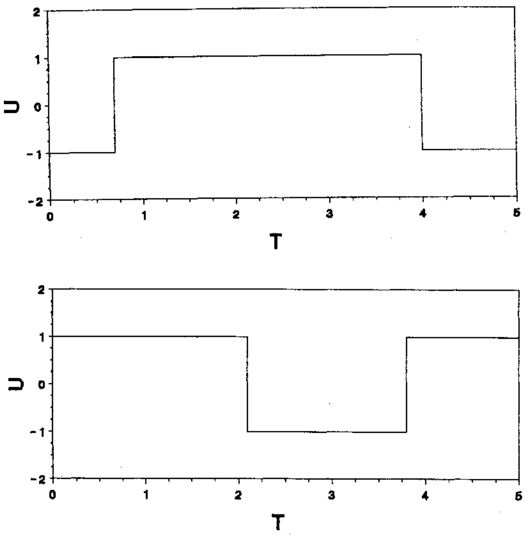

Fig. 3 Control options for the constrained problem.

where c is an integration constant. This result is valid for our problem only if

$$|\gamma_e(t_f)| = \dot{\gamma}_{e_m} \left| \int_0^{t_f} \text{sgn}\hat{\lambda}_1(t)\, dt \right| < \gamma_f \tag{29}$$

We now consider the case in which the terminal path-angle constraint is binding and note that the solution for the last three adjoint equations is unchanged (since this set is independent of the first co-state variable). However, the first co-state (the switching function) is now

$$\lambda_1(\theta) = \hat{\lambda}_1(\theta) + \nu \tag{30}$$

where ν will be determined by

$$-\int_0^{t_f} \text{sgn}(\hat{\lambda}_1 + \nu)\, dt = \frac{\gamma_f}{\dot{\gamma}_{e_m}} \tag{31}$$

Consequently, the solution contains up to $N' - 1$ switching points that are the zeros of λ_1. Since these zeros determine its integrand, the last integral is continuous with ν (from the continuous behavior of the roots) but is not, in general, differentiable. Consequently, the implicit function theorem is not applicable and the solution for ν may not be unique.

In Fig. 3, two alternatives for the solution are presented ($N' = 3$ and T is the elapsed time). Both of them satisfy the terminal constraint and are locally optimal; however, only one of these is the global optimum. The other solution can be eliminated by the terminal sign of λ_1 [i.e., the sign of $u(t)$, as required by the transversality conditions].

Computational Results and Problem Decomposition

The solutions for the two-point boundary-value problem were obtained numerically using a multiple shooting algorithm (MSA).[6] The initial states are all zero and the following numerical values are used:

$$N' = 3$$

$$V'_p = 1000 \text{ fps}$$

$$V'_e = 500 \text{ fps}$$

$$\dot{\gamma}_{e_m} = 0.3^1/\text{s}$$

$$\tau_p = 0.5 \text{ s}$$

The problem was solved for two terminal terms: $t_f = 10\tau_p$ and $t_f = 6\tau_p$. These data correspond, for example, to a tail-chase conflict ($\gamma_{p_o} = \gamma_{e_o} = 0$) with initial distances (PE) of 2500 ft and 1500 ft, respectively. Figure 4 presents the miss distance (normalized by $V'_e\tau^2\dot{\gamma}_{e_m}$) as a function of the terminal constraint, and Fig. 5 presents the switching points of the optimal control function for those values.

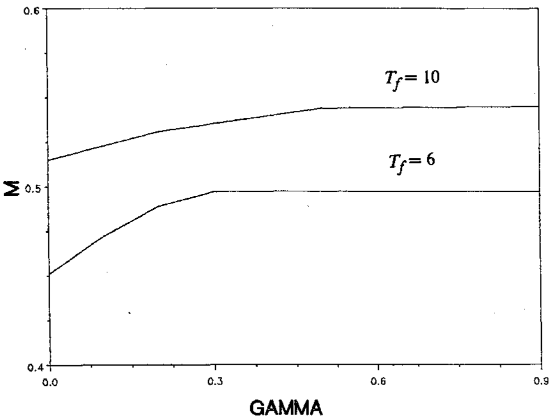

Fig. 4 Miss distance.

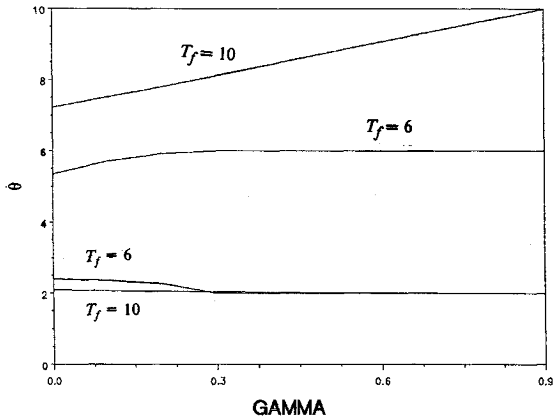

Fig. 5 Switching points.

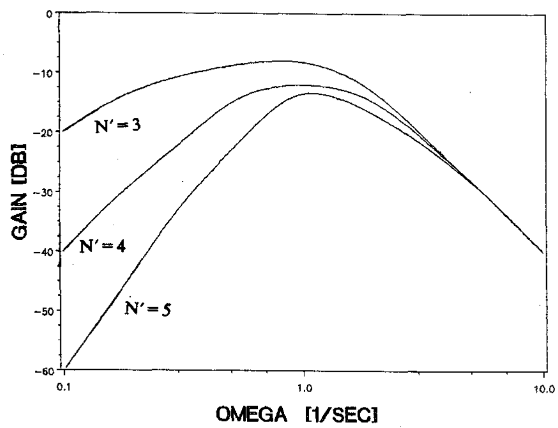

Fig. 6 Miss distance influence function.

As it is shown in Fig. 5 for $t_f \gg \tau_p$, the original switching point hardly moves as the constraint is tightened, and the additional switching point is the one that is responsible for satisfying the path-angle constraint. This fact is true in general for any N'.

To explain this interesting phenomenon, we shall present two considerations: The first will be purely mathematical and the second will be of a more physical and heuristic type.

The adjoint system analysis, as indicated in Ref. 4, interprets $\hat{\lambda}_1$ as the influence function of the control u on the miss distance. From the Bode representation of $\hat{\lambda}_1$ (Fig. 6), obtained by

Eq. (28), we conclude that it behaves as a bandpass filter; hence, the low-frequency content of the control has little effect on the payoff. The path angle is an integral of the control and is affected by the control history from t_0 onward (as a low-pass filter). Thus, in the optimal control, the constraint is satisfied by appropriately selecting the early switching point. Altering the constraint causes the early switching point to move but has little effect on the late switches.

The physical interpretation is more obvious. The basic principle of the evasion maneuver, under our assumptions, is to take advantage of the finite time lag of P; hence, only the last few time constants are of importance. E, however, knows "the future" [assumption (5)] so it can adjust the terminal path angle to the constraint by adding the initial switching point without significantly affecting the payoff.

As a result, the constrained solution can be composed from the optimal unconstrained one, plus one more switching point to satisfy the constraint. Unless t_f is of the same order as τ_p, this will be a good approximation for the solution. Moreover, we can conclude from Fig. 8 that this decomposition is more valid for larger values of N'.

Optimal Evasion Against Two (or More) Pursuers

Problem Formulation

We shall consider the problem of optimal evasion against two guided missiles under the simplifying assumptions of the previous section for each of the pursuers and the evader. Thus, each pursuer constructs its own collision triangle around which we may linearize the kinematics. For simplicity, we will further assume that the pursuers are identical, i.e., they have the same time delay τ_p and the same gain N'.

Let Δ denote the nominal time difference between impacts, P_2 be the first of the two pursuers, and the state vector be

$$x = \mathrm{col}(\gamma_e, y_1, \gamma_{p_1}, \dot{\gamma}_{p_1}, y_2, \gamma_{p_2}, \dot{\gamma}_{p_2}) \tag{32}$$

and the control u

$$u = \frac{\dot{\gamma}_e}{\dot{\gamma}_{e_m}} \tag{33}$$

The linear state equations relating x to u are

$$\dot{x}_1 = \dot{\gamma}_{e_m} u \tag{34}$$

$$\dot{x}_2 = V'_e x_1 - V'_{p_1} x_3 \tag{35}$$

$$\dot{x}_3 = x_4 \tag{36}$$

$$\dot{x}_4 = \frac{N' V'_e}{\tau_p V'_{p_1}(t_f - t)} x_1 + \frac{N'}{\tau_p V'_{p_1}(t_f - t)^2} x_2 - \frac{N'}{\tau_p (t_f - t)} x_3 - \frac{1}{\tau_p} x_4 \tag{37}$$

$$\dot{x}_5 = (V'_e x_1 - V'_{p_2} x_6) h(t_f - \Delta - t) \tag{38}$$

$$\dot{x}_6 = (x_7) h(t_f - \Delta - t) \tag{39}$$

$$\dot{x}_7 = \frac{N' V'_e}{\tau_p V'_{p_2}(t_f - t - \Delta)} h(t_f - \Delta - t) x_1 + \frac{N'}{\tau_p V'_{p_2}(t_f - t - \Delta)^2} h(t_f - \Delta - t) x_5 - \frac{N'}{\tau_p (t_f - t - \Delta)} h(t_f - \Delta - t) x_6 - \frac{1}{\tau_p} h(t_f - \Delta - t) x_7 \tag{40}$$

When $h(z)$ is defined by the Heaviside step function,

$$h(z) = \begin{cases} 1 & \text{if} \quad z \geq 0 \\ 0 & \text{if} \quad z < 0 \end{cases} \tag{41}$$

Hence, after P_2 finishes its mission, only the first four equations remain active. It is noted that now we have two singular points (i.e., the nominal collision points) that should be taken into account during any numerical integration of the state equations.

The optimal control problem is to find u that minimizes $J = -y_1^2(t_f)$, subject to $|u| \leq 1$ and $y_2^2(t_f - \Delta) = c$, when $0 \leq c \leq y_m^2$ and y_m^2 is the optimal performance index in the case of the single pursuer. Consequently, a simple survey over the parameter c will provide the solution for the problem of maximizing the smaller of the two miss distances in our case.

The time difference Δ will also be a parameter that may be adjusted by the pursuers to maximize J. (We may think of two air-to-air missiles launched by the same aircraft.)

Problem Analysis

We shall define the Hamiltonian to be

$$H(\lambda, x, u) = \dot{x}_1\lambda_1 + \dot{x}_2\lambda_2 + \dot{x}_3\lambda_3 + \dot{x}_4\lambda_4 + \dot{x}_5\lambda_5 + \dot{x}_6\lambda_6 + \dot{x}_7\lambda_7 \tag{42}$$

and introduce, as before, the nondimensional time to go

$$\theta = \frac{(t_f - t)}{\tau_p} \tag{43}$$

We shall redefine Δ to be in τ_p units. The adjoint system is

$$\frac{\mathrm{d}\lambda_i(\theta)}{\mathrm{d}\theta} = \tau_p \frac{\partial H}{\partial x_i} \tag{44}$$

The transversality conditions for our case are

$$\lambda_i(0) = 0 \qquad \text{for} \qquad i \neq 2 \tag{45}$$

$$\lambda_2(0) = -2y_1(0) \tag{46}$$

At the time of the first nominal impact $\theta = \Delta$, we shall have the following conditions:

$$\lambda_i(\Delta+) = \lambda_i(\Delta-) \qquad \text{for} \qquad i \neq 5 \tag{47}$$

$$\lambda_5(\Delta+) = \lambda_5(\Delta-) + \nu \tag{48}$$

From the maximum principle, as in the preceding sections, the first adjoint parameter λ_1 is the switching function. It can be shown that[7]

$$\lambda_1(\theta) = c_1 p(\theta) \exp(-\theta) + c_2 p(\theta - \Delta) \exp(-\theta + \Delta) h(\theta - \Delta) \tag{49}$$

A complete solution for the co-state vector when $N' = 3$ and $V'_{p_1} = V'_{p_2} = V_p$ is given in Ref. 7.

We may draw now two conclusions regarding the switching points. First, there will be no more than $(2N' - 3)$ of them, and the second conclusion is that any switching point of the single pursuer case that takes place after $t_f - \Delta\tau_p$ also will be a switching point for our present case. Both the conclusions may be verified by checking the roots of λ_1.

We can use now a simple parameter optimization procedure to find out the exact location of the switching points, from which the miss distance may be obtained, as suggested in Ref. 4, by the simple relation

$$y_1(t_f) = \tau_p^2 \ddot{y}_{e_m} \left[g(\theta_0) + 2 \sum_i g(\theta_i) \right] \tag{50}$$

where $\ddot{y}_{em} = \dot{\gamma}_{em} V'_e$ and

$$g(x) = \int_0^x \hat{\lambda}_1(x')\,\mathrm{d}x' \tag{51}$$

This relation is based on the observation made about $\hat{\lambda}_1$ (the solution for the single pursuer case) as the impulse response

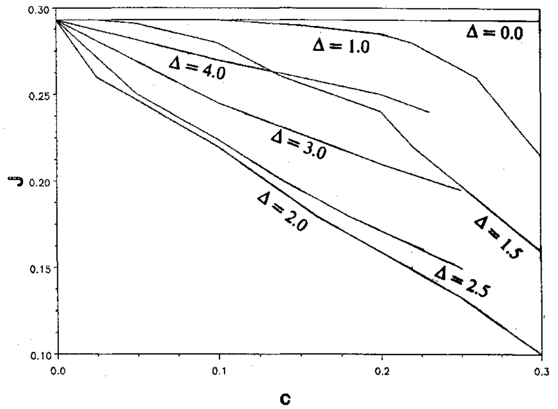

Fig. 7 Payoff vs constraint.

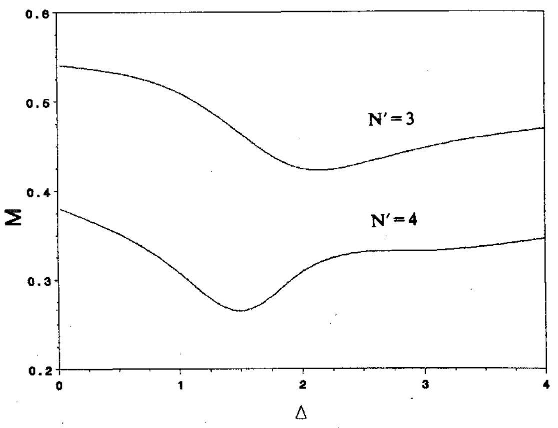

Fig. 8 Payoff vs nominal time between impacts.

function relating control to miss distance. In general, as indicated in Ref. 4, the first term in the foregoing expression for the miss distance can be neglected because of the strong damping of $\exp(-\theta_0)$.

Computational Results

The solution can be completed by employing a parameter optimization technique to locate the switching points. To illustrate the procedure, let $N' = 3$; thus, the payoff to be maximized is

$$J = \left[\sum_{i=1}^{3} (-1)^i \exp(-\theta_i)\theta_i^2 \right]^2 \qquad (52)$$

where θ_i is the ith switching point and J the square of the miss distance of P_1. However, this should be subjected to a constraint on the other miss distance

$$\left[\sum_{i=1}^{3} (-1)^i \exp(\Delta - \theta_i)(\Delta - \theta_i)^2 h(\theta_i - \Delta) \right]^2 - c \geq 0 \qquad (53)$$

Notice that both the payoff and the constraint are analytically differentiable. Figure 7 presents the results (payoff vs constraint) for various Δ as obtained by an optimization program that employs a gradient projection algorithm.[8] The case $\Delta = 0$ is a limit case in which the pursuers coincide with each other. The optimal switching points as obtained by the program are in agreement with the prediction of the previous section.

As expected, the payoff is changed monotonically with the constraint. In order to maximize the closest approach of the pursuers, we need to take the results from the 45 deg line on which the two miss distances are the same.

Figure 8 presents the optimal miss distance against Δ. Notice the existence of an optimum Δ, from the pursuer point of view, for which the maximal miss distance is minimized. The results for $N' = 4$ have been obtained in the same way and a different optimal Δ was found as illustrated in Fig. 8. For more than two pursuers, the analysis is virtually the same, and we end up with $[n(N' - 1) - 1]$ as the maximal number of switching points in the case of n pursuers.

Conclusions

By applying linearized kinematics to the optimal evasion problem, solutions for some relatively complicated cases have been obtained. The optimal commanded lateral acceleration is, in general, a bang-bang nonsingular one governed by a switching function. The number of switching points and their location is dependent on the dynamics of the pursuer and the evader, and the particular constraints of the problem in hand.

We emphasize again the limitations of the linearized planar constant speed motion that has been assumed in this work.[7] For the unconstrained basic evasion problem, a three-dimensional analysis is found in Ref. 3 and some nonlinear effects have been investigated in Ref. 9. The variable speed case is characterized by state-dependent control constraints that may significantly change the nature of the solution and is recommended for future investigation.

Acknowledgment

This research was supported, in part, by the U.S. Air Force and U.S. Navy under Contract N60921-83G-A165.

References

[1]Julich, P. M. and Borg, D. A., "Proportional Navigation vs an Optimally Evading Constant-Speed Target in Two Dimensions," *Journal of Spacecraft and Rockets,* Vol. 7, Dec. 1970, pp. 1454–1457.

[2]Slater, G. L. and Wells, W. R., "Optimal Evasive Tactics with Time Delay," *Journal of Spacecraft and Rockets,* Vol. 10, May 1973, pp. 309–313.

[3]Shinar, J., Rosenzstein, Y., and Bezner, E., "Analysis of Three-Dimensional Optimal Evasion with Linearized Kinematics," *Journal of Guidance and Control,* Vol. 2, Sept. 1979, pp. 353–360.

[4]Shinar, J. and Steinberg, D., "Analysis of Optimal Evasive Maneuvers Based on a Linearized Two-Dimensional Model," *Journal of Aircraft,* Vol. 14, Aug. 1977, pp. 795–802.

[5]Bryson, A. E. and Ho, Y. C., *Applied Optimal Control,* Hemisphere, Washington, DC, 1969.

[6]Bulirsch, R., "Die Mehrzielmethode zur numerischen Lösung von nichtlinearen Randwertproblemen und Aufgaben der optimalen Steuerung," Rept. of Carl Cranz Gesellschaft, Oberpfaffenhofen, FRG, Oct. 1971.

[7]Ben-Asher, J., "Optimal Evasion Against a Proportionally Guided Pursuer," M.S. Thesis, Virginia Polytechnic Inst. and State Univ., Blacksburg, VA, 1986.

[8]Kelley, H. J. and Speyer, J. L., "Accelerated Gradient Projection," *Lecture Notes on Mathematics 132,* Springer-Verlag, Berlin, 1970.

[9]Forte, I., Steinberg, A., and Shinar, J., "The Effect of Non-Linear Kinematics in Optimal Evasion," *Optimal Control Applications and Methods,* Vol. 4, April–June 1983, pp. 139–152.